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Complete and exact solutions of a penny-shaped crack in a piezoelectric solid: antisymmetric shear loadings

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Abstract

This paper presents an exact analysis of the problem of a penny-shaped crack in a transversely isotropic piezoelectric medium subjected to arbitrary shear loading that is antisymmetric with respect to the crack plane. The analysis is based on the general solution of three-dimensional piezoelasticity, which is represented by four quasi-harmonic displacement functions. It is shown that these harmonics can be represented by one complex potential. By using the previous results in potential theory, an exact solution is obtained. In particular, for uniform shear and point shear loadings, complete expressions for the elastoelectric field are derived in terms of elementary functions. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Piezoelectric materials play key roles as active components in technologies such as infranics, navigation, and electrics, etc. because of their particular coupling effect between elastic deformation and electric field. Due to the inherent brittle weakness of piezoelectric ceramics, the fracture of piezoelectric materials has gained considerable interest. Most available theoretical works are concerned with the two-dimensional study of cracks in piezoelectric materials, see Pak (1990), Sosa (1992), Suo et al. (1992), Park and Sun (1995), Zhang and Tong (1996), Zhong and Meguid (1997), etc. There are comparatively few works of three-dimensional analysis (Deeg, 1980; Sosa and Pak, 1990; Wang, 1992; Huang, 1997; Chen and

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Shioya, 1999). In particular, for a penny-shaped crack in a transversely isotropic piezoelectric medium, Kogan et al. (1996) derived the exact solutions for axisymmetric and antisymmetric far field uniform loadings from those of a spheroidal piezoelectric inclusion as a limiting case.

It is noted that for transversely isotropic elastic materials, significant results have been newly developed by Fabrikant (1989) for the application of potential theory in analyzing contact and crack problems in elasticity. Using the new results, Fabrikant (1989) has showed that an exact solution of a penny-shaped crack can be obtained for antisymmetric shear loading. In particular, for both uniform and point shear loadings, complete solutions can be derived in terms of elementary functions.

In this paper, we intend to use the results of Fabrikant (1989) to analyze a penny-shaped crack in a transversely isotropic piezoelectric medium subjected to arbitrary shear loading that is antisymmetrically applied to the upper and lower crack faces. To this end, the recently proposed general solution to the coupled equations for transversely isotropic piezoelectric solids is employed (Ding et al., 1996, 1997). The general solution is expressed in terms of four quasi harmonics, which can be represented only by one complex potential. It will be shown that the satisfaction of the boundary conditions finally leads to an integro–differential equation that has an identical structure to the one for elasticity. The only difference is the definition of the involved material constants, which has no effect on the form of the solution. Thus, the results presented in Fabrikant (1989) can be utilized to obtain the exact solution of the problem. Especially, for two loading cases, i.e. uniform shear and point shear loadings, complete solutions are exactly derived in terms of elementary functions. The present results for the uniform loading are compared with those available in the literature (Kogan et al., 1996) and good agreement is observed.

It is also noted here that by using the recent results presented in Fabrikant (1996a, 1996b), a complete solution can also be obtained for an external circular crack in a transversely isotropic medium subjected to arbitrary shear loading.

2. Basic equations for piezoelasticity and the general solution

The piezoelastic governing equations of a transversely isotropic piezoelectric medium can be found in Tiersten (1969). In Cartesian coordinates (with the z-axis being normal to the plane of isotropy, i.e. the x–y plane), these equations can be rewritten in a complex form, by introducing the tangential complex displacement $U = u + \mathrm{i}v$, as follows,

$$\frac{1}{2}(c_{11}+c_{66})\Delta U + c_{44}\frac{\partial^2 U}{\partial z^2} + \frac{1}{2}(c_{11}-c_{66})\Lambda^2\bar{U} + (c_{13}+c_{44})\Lambda\frac{\partial w}{\partial z} + (e_{15}+e_{31})\Lambda\frac{\partial \Phi}{\partial z} = 0,$$

$$\frac{1}{2}(c_{13}+c_{44})\frac{\partial}{\partial z}(\bar{\Lambda}U + \Lambda\bar{U}) + c_{44}\Delta w + c_{33}\frac{\partial^2 w}{\partial z^2} + e_{15}\Delta\Phi + e_{33}\frac{\partial^2\Phi}{\partial z^2} = 0$$

and

$$\frac{1}{2}(e_{15}+e_{31})\frac{\partial}{\partial z}(\bar{\Lambda}U + \Lambda\bar{U}) + e_{15}\Delta w + e_{33}\frac{\partial^2 w}{\partial z^2} - \epsilon_{11}\Delta\Phi - \epsilon_{33}\frac{\partial^2\Phi}{\partial z^2} = 0, \tag{1}$$

where, $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, $\Lambda = \partial/\partial x + \mathrm{i}\partial/\partial y$, and the overbar indicates the complex conjugate value. $(u, v, w)^T$ and Φ are displacement vector and electric potential, respectively. c_{ij}, ϵ_{ij} and e_{ij} are the elastic, dielectric, and piezoelectric constants, respectively. The general solution of Eq. (1) proposed by Ding et al. (1997) is also rewritten in a complex form:

$$U = \Lambda\left(\sum_{i=1}^{3} F_i + \mathrm{i}F_4\right),$$

$$w = \sum_{i=1}^{3} \alpha_{i1} \frac{\partial F_i}{\partial z_i}$$

and

$$\Phi = \sum_{i=1}^{3} \alpha_{i2} \frac{\partial F_i}{\partial z_i}, \tag{2}$$

where:

$$\alpha_{i1} = \frac{c_{11}\epsilon_{11} - m_3 s_i^2 + c_{44}\epsilon_{33} s_i^4}{(m_1 - m_2 s_i^2) s_i},$$

$$\alpha_{i2} = \frac{c_{11}e_{15} - m_4 s_i^2 + c_{44}e_{33} s_i^4}{(m_1 - m_2 s_i^2) s_i},$$

$$m_1 = \epsilon_{11}(c_{13} + c_{44}) + e_{15}(e_{15} + e_{31}),$$

$$m_2 = \epsilon_{33}(c_{13} + c_{44}) + e_{33}(e_{15} + e_{31}),$$

$$m_3 = c_{11}\epsilon_{33} + c_{44}\epsilon_{11} + (e_{15} + e_{31})^2,$$

$$m_4 = c_{11}e_{33} + c_{44}e_{15} - (c_{13} + c_{44})(e_{15} + e_{31}), \tag{3}$$

and $z_i = s_i z$, $s_4^2 = c_{66}/c_{44}$ and s_i^2 (i = 1, 2, 3) are roots of the following algebraic equation:

$$as^6 - bs^4 + cs^2 - d = 0, \tag{4}$$

where,

$$a = c_{44}\left(e_{33}^2 + c_{33}\epsilon_{33}\right),$$

$$b = c_{33}m_3 + \epsilon_{33}\left[c_{44}^2 - (c_{13} + c_{44})^2\right] + e_{33}(2m_4 - c_{11}e_{33}),$$

$$c = c_{44}m_3 + \epsilon_{11}\left[c_{11}c_{33} - (c_{13} + c_{44})^2\right] + e_{15}(2m_4 - c_{44}e_{15})$$

and

$$d = c_{11}\left(e_{15}^2 + c_{44}\epsilon_{11}\right), \tag{5}$$

It is noted here that the general solution given in Eq. (2) is only valid for distinct s_i^2 while, for other cases, different forms should be adopted (Ding et al., 1997; Chen, 2000), see Appendix A. It is also required that $F_i(z)$ should satisfy the following quasi harmonic equation

$$\left(\Lambda + \frac{\partial^2}{\partial z_i^2}\right)F_i = 0,\ (i = 1,\ 2,\ 3,\ 4). \tag{6}$$

By virtue of the linear constitutive relations of a piezoelectric body [see Eq. (1) in Ding et al. (1997)], the following expressions for stresses $\sigma_i(\tau_{ij})$ and electric displacements D_i can be derived:

$$\sigma_1 = 2\sum_{i=1}^{3}\frac{\partial^2}{\partial z_i^2}[(c_{66} - c_{11}) + c_{13}s_i\alpha_{i1} + e_{31}s_i\alpha_{i2}]F_i,$$

$$\sigma_2 = 2c_{66}\Lambda^2\left(\sum_{i=1}^{3}F_i + \mathrm{i}F_4\right),$$

$$\sigma_z = \sum_{i=1}^{3}\frac{\partial^2}{\partial z_i^2}\gamma_{1i}F_i,$$

$$\tau_z = \Lambda\left(\sum_{i=1}^{3}\gamma_{1i}s_i\frac{\partial}{\partial z_i}F_i + \mathrm{i}s_4c_{44}\frac{\partial}{\partial z_4}F_4\right),$$

$$D = \Lambda\left(\sum_{i=1}^{3}\gamma_{2i}s_i\frac{\partial}{\partial z_i}F_i + \mathrm{i}s_4e_{15}\frac{\partial}{\partial z_4}F_4\right),$$

$$D_z = \sum_{i=1}^{3}\frac{\partial^2}{\partial z_i^2}\gamma_{2i}F_i, \tag{7}$$

where, $\sigma_1 = \sigma_x + \sigma_y$, $\sigma_2 = \sigma_x - \sigma_y + 2\mathrm{i}\tau_{xy}$, $\tau_z = \tau_{xz} + \mathrm{i}\tau_{yz}$ and $D = D_x + \mathrm{i}D_y$. In addition,

$$\gamma_{1i} = -c_{13} + c_{33}s_i\alpha_{i1} + e_{33}s_i\alpha_{i2}$$

and

$$\gamma_{2i} = -e_{31} + e_{33}s_i\alpha_{i1} - \epsilon_{33}s_i\alpha_{i2}.$$

It is noted that the following identities have been employed in Eq. (7):

$$\gamma_{1i}s_i = c_{44}(s_i + \alpha_{i1}) + e_{15}\alpha_{i2}$$

and

$$\gamma_{2i}s_i = e_{15}(s_i + \alpha_{i1}) + \epsilon_{11}\alpha_{i2}. \tag{8}$$

3. The potential theory method for antisymmetric crack problem

It is firstly considered that a transversely isotropic piezoelectric solid of infinite extent is weakened by a flat crack S in the plane $z = 0$. The crack is subjected to arbitrary shear loading that is antisymmetric with respect to the crack plane. The problem can, in fact, be described as a mixed boundary value problem of a half-space $z \geq 0$, with the following boundary conditions on the plane $z = 0$:

$$\tau_z = -\tau(x,y), \text{ for } (x,y) \in S,$$

$$U = 0, \text{ for } (x,y) \notin S$$

and

$$\sigma_z = D_z = 0, \text{ for } -\infty<(x,y)<\infty. \tag{9}$$

Similar to the pure elasticity (Fabrikant, 1989), Eq. (9) can be satisfied by a representation of the general solution in terms of one complex harmonic function F, namely,

$$F_i(z) = c_i\left[\Lambda \bar{F}(z_i) + \bar{\Lambda} F(z_i)\right], \ (i = 1, 2, 3)$$

and

$$F_4(z) = c_4\left[\Lambda \bar{F}(z_4) - \bar{\Lambda} F(z_4)\right], \tag{10}$$

where c_i ($i = 1, 2, 3, 4$) are undetermined constants and

$$F(\rho,\phi,z) = \int\int_S \ln[R(M,N) + z]U(N)\mathrm{d}S_N, \tag{11}$$

where $R(M,N)$ is the distance between two points $M(\rho,\phi,z)$ and $N(r,\psi,0)$, $N \in S$, and the integration is taken over the crack domain S. Hereafter, the cylindrical coordinates (ρ,ϕ,z) are alternatively used for the sake of convenience. By assuming

$$\sum_{i=1}^{3} c_i\gamma_{1i} = 0$$

and

$$\sum_{i=1}^{3} c_i\gamma_{2i} = 0, \tag{12}$$

the third condition in Eq. (9) is identically satisfied. It can be further verified that the second condition in Eq. (9) can be satisfied if

$$\sum_{i=1}^{3} c_i + \mathrm{i}c_4 = 0$$

and

$$\sum_{i=1}^{3} c_i - \mathrm{i}c_4 = \frac{1}{2\pi}. \tag{13}$$

The combination of Eqs. (12) and (13) yields:

$$\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \end{Bmatrix} = \frac{1}{4\pi}\begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ 1 & 1 & 1 \end{bmatrix}^{-1}\begin{Bmatrix} 0 \\ 0 \\ 1 \end{Bmatrix},\ c_4 = \frac{i}{4\pi}. \tag{14}$$

Substituting Eq. (10) into Eq. (7) gives the expression of τ_z for $z = 0$ as

$$\tau_z\big|_{z=0} = \left(\sum_{i=1}^{3} c_i\gamma_{1i}s_i - \frac{s_4c_{44}}{4\pi}\right)\Lambda^2\frac{\partial}{\partial z}\bar{F}(z) + \left(\sum_{i=1}^{3} c_i\gamma_{1i}s_i + \frac{s_4c_{44}}{4\pi}\right)\Lambda\frac{\partial}{\partial z}F(z). \tag{15}$$

Now noticing the first condition in Eq. (9), the following integro–differential equation is obtained:

$$\tau(N_0) = -\frac{1}{2\pi^2\left(G_1^2 - G_2^2\right)}\left[G_1\Lambda\int\int_S \frac{U(N)}{R(N,N_0)}\mathrm{d}S_N + G_2\Lambda^2\int\int_S \frac{\bar{U}(N)}{R(N,N_0)}\mathrm{d}S_N\right], \tag{16}$$

where N_0, $N \in S$ and

$$G_1 = \beta + H$$

$$G_2 = \beta - H,$$

$$\beta = \frac{1}{2\pi s_4 c_{44}}$$

and

$$H = \frac{1}{8\pi^2\sum_{i=1}^{3} c_i\gamma_{1i}s_i}. \tag{17}$$

Here, two new material constants G_1 and G_2 are introduced so as to make the resulting integro–differential equation Eq. (16) have exactly the same form as that for elasticity (Fabrikant, 1989). Such an equation has been solved for a penny-shaped crack in transversely isotropic elastic media. Therefore, the results obtained by Fabrikant (1989) can be used to obtain the exact and complete solutions for a penny-shaped crack in piezoelectric materials.

4. The complete solutions for a penny-shaped crack

In the case that the crack is penny shaped, utilizing the results of Fabrikant (1989), we can directly write down the solution of Eq. (16) as

$$U(\rho,\phi,0) = \frac{G_1}{\pi}\int_0^{2\pi}\int_0^{a}\left[\frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right) - \frac{G_2^2}{G_1^2}\frac{(3-\bar{t})\eta}{a^2(1-\bar{t})^2}\right]\tau(\rho_0,\phi_0)\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0$$
$$+\frac{G_2}{\pi}\int_0^{2\pi}\int_0^{a}\left[\frac{q}{R\bar{q}}\tan^{-1}\left(\frac{\eta}{R}\right) + \frac{\eta(q/\bar{q} - t\mathrm{e}^{\mathrm{i}2\phi_0})}{a^2(1-t)(1-\bar{t})}\right]\bar{\tau}(\rho_0,\phi_0)\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0, \tag{18}$$

where a is the radius of the crack and

$$R = \left[\rho^2 + \rho_0^2 - 2\rho\rho_0\cos(\phi - \phi_0)\right]^{1/2},$$

$$\eta = \frac{(a^2 - \rho^2)^{1/2}\left(a^2 - \rho_0^2\right)^{1/2}}{a},$$

$$q = \rho\mathrm{e}^{\mathrm{i}\phi} - \rho_0\mathrm{e}^{\mathrm{i}\phi_0}$$

and

$$t = \left(\frac{\rho\rho_0}{a^2}\right)\mathrm{e}^{\mathrm{i}(\phi-\phi_0)}. \tag{19}$$

By substituting Eq. (18) into Eq. (11) and these, in turn, into Eqs. (2) and (7), exact expressions for the elastoelectric field can be derived. It is noted that for the point loading and the uniform loading cases, Fabrikant (1989) has derived the elastic solutions in terms of elementary functions. Similar to his derivations, we can also obtain the corresponding ones for piezoelectric materials. Details are however omitted, and the solutions are given in the following.

4.1. Uniform shear loading

Denote the uniform shear loading as $\tau_0 = \tau_{xz}^0 + \mathrm{i}\tau_{yz}^0$, where τ_{xz}^0 and τ_{yz}^0 are constants. By virtue of the results presented in Fabrikant (1989), we obtain the following expressions for the elastoelectric field:

$$U = \frac{4\beta H}{G_1}\left\{\sum_{i=1}^{3} C_i\left[g_1(z_i)\tau_0 + g_2(z_i)\tau_0\right] - \left[g_1(z_4)\bar{\tau}_0 + g_2(z_4)\tau_0\right]\right\},$$

$$w = \frac{2\beta H}{G_1}\rho\left(\bar{\tau}_0\mathrm{e}^{\mathrm{i}\phi} + \tau_0\mathrm{e}^{-\mathrm{i}\phi}\right)\sum_{i=1}^{3} C_i\alpha_{i1}\left[\frac{a\left(l_{2i}^2 - a^2\right)^{1/2}}{l_{2i}^2} - \sin^{-1}\left(\frac{a}{l_{2i}}\right)\right],$$

$$\Phi = \frac{2\beta H}{G_1}\rho\left(\bar{\tau}_0\mathrm{e}^{\mathrm{i}\phi} + \tau_0\mathrm{e}^{-\mathrm{i}\phi}\right)\sum_{i=1}^{3} C_i\alpha_{i2}\left[\frac{a\left(l_{2i}^2 - a^2\right)^{1/2}}{l_{2i}^2} - \sin^{-1}\left(\frac{a}{l_{2i}}\right)\right],$$

$$\sigma_1 = \frac{8\beta H}{G_1}\left(\bar{\tau}_0\mathrm{e}^{\mathrm{i}\phi} + \tau_0\mathrm{e}^{-\mathrm{i}\phi}\right)\sum_{i=1}^{3} C_i[(c_{66} - c_{11}) + c_{13}s_i\alpha_{i1} + e_{31}s_i\alpha_{i2}]g_3(z_i),$$

$$\sigma_2 = -\frac{8c_{66}\beta H e^{i\phi}}{G_1}\left\{\sum_{i=1}^{3} C_i\left[g_4(z_i)\bar{\tau}_0 + g_3(z_i)\tau_0\right] - \left[g_4(z_4)\bar{\tau}_0 - g_3(z_4)\tau_0\right]\right\}$$

$$\sigma_z = \frac{4\beta H}{G_1}\left(\bar{\tau}_0 e^{i\phi} + \tau_0 e^{-i\phi}\right)\sum_{i=1}^{3} C_i\gamma_{1i}g_3(z_i),$$

$$\tau_z = \frac{4\beta H}{G_1}\left\{\sum_{i=1}^{3} C_i\gamma_{1i}s_i\left[g_5(z_i)\bar{\tau}_0 + g_6(z_i)\tau_0\right] - s_4c_{44}\left[g_5(z_4)\bar{\tau}_0 - g_6(z_4)\tau_0\right]\right\}$$

$$D = \frac{4\beta H}{G_1}\left\{\sum_{i=1}^{3} C_i\gamma_{2i}s_i\left[g_5(z_i)\bar{\tau}_0 + g_6(z_i)\tau_0\right] - s_4e_{15}\left[g_5(z_4)\bar{\tau}_0 + g_6(z_4)\bar{\tau}_0\right]\right\}$$

and

$$D_z = \frac{4\beta H}{G_1}\left(\bar{\tau}_0 e^{i\phi} + \tau_0 e^{-i\phi}\right)\sum_{i=1}^{3} C_i\gamma_{2i}g_3(z_i), \tag{20}$$

where $C_i = 4\pi c_i$, $l_{1;2i} = \{[(\rho+a)^2 + z_i^2]^{1/2} \mp [(\rho-a)^2 + z_i^2]^{1/2}\}/2$ and

$$g_1(z) = e^{2i\phi}\frac{\left(l_1^2 + 2a^2\right)\left(a^2 - l_1^2\right)^{1/2} - 2a^3}{3\rho^2},$$

$$g_2(z) = \left(a^2 - l_1^2\right)^{1/2} - z\sin^{-1}\left(\frac{a}{l_2}\right),$$

$$g_3(z) = \frac{al_1\left(a^2 - l_1^2\right)^{1/2}}{l_2\left(l_2^2 - l_1^2\right)},$$

$$g_4(z) = g_3(z)e^{2i\phi} + 4g_1(z)/\rho,$$

$$g_5(z) = e^{2i\phi}\frac{al_1^2\left(l_2^2 - a^2\right)^{1/2}}{l_2^2\left(l_2^2 - l_1^2\right)}$$

and

$$g_6(z) = -\sin^{-1}\left(\frac{a}{l_2}\right) + \frac{a\left(l_2^2 - a^2\right)^{1/2}}{l_2^2 - l_1^2}. \tag{21}$$

Here $l_{1;2} = \{[(\rho+a)^2 + z^2] \mp [(\rho-a)^2 + z^2]^{1/2}\}/2$.

The expressions presented in Eq. (20) degenerate identically to those for elasticity if the electric effect is neglected. However, there are some misprints in Fabrikant (1989): factor 2 should be dropped in Eq.

(4.6.2), the coefficient in Eq. (4.6.3) should read $m_k/(m_k\gamma_k - \gamma_k)$ rather than $m_k/(m_k - 1)$, and the second term of the right side of Eq. (4.6.10) should be divided by ρ.

Defining the complex stress intensity factor, as follows

$$K_{\mathrm{II}} + \mathrm{i}k_{\mathrm{III}} = \lim_{\rho \to a} \left\{ (\rho - a)^{1/2} \mathrm{e}^{-\mathrm{i}\phi} \tau_z \big|_{z=0} \right\}, \tag{22}$$

where k_{II} and k_{III} actually correspond to the mode II and mode III intensity factors of a penny-shaped crack, respectively. Noticing the following property:

$$l_{1i} \to \min\,(a,\rho) \text{ and } l_{2i} \to \max\,(a,\rho), \text{ when } z = 0, \tag{23}$$

Eq. (20) gives τ_z at $z = 0$ for $\rho > a$:

$$\tau_z\big|_{z=0} = \frac{2}{\pi G_1} \left\{ G_1 \left[\frac{a}{(\rho^2 - a^2)^{1/2}} - \sin^{-1}\left(\frac{a}{\rho}\right) \right] \tau_0 + G_2 \frac{a^3 \mathrm{e}^{\mathrm{i}2\phi}}{\rho^2 (\rho^2 - a^2)^{1/2}} \bar{\tau}_0 \right\}. \tag{24}$$

Thus, one obtains,

$$k_{\mathrm{II}} + \mathrm{i}k_{\mathrm{III}} = \frac{\sqrt{2a}}{\pi} \left(\mathrm{e}^{-\mathrm{i}\phi} \tau_0 + \frac{G_2}{G_1} \mathrm{e}^{\mathrm{i}\phi} \bar{\tau}_0 \right). \tag{25}$$

4.2. Point shear loading

It is now supposed that the crack is subjected to a pair of concentrated shear forces $T = T_x + \mathrm{i}T_y$ that are applied to the crack faces antisymmetrically at the points $(\rho_0,\ \phi_0,\ 0^{\pm})$, $\rho_0 < a$. Using Fabrikant's results (Fabrikant, 1989), we obtain the following expressions for the elastoelectric field:

$$U = \frac{H}{\pi} \sum_{i=1}^{3} C_i \left\{ \left[f_2(z_i) + \frac{G_2}{G_1} \bar{f}_7(z_i) \right] T - \left[f_{16}(z_i) + \frac{G_2}{G_1} f_8(z_i) \right] \bar{T} \right\}$$
$$+ \frac{\beta}{\pi} \left\{ \left[f_2(z_4) - \frac{G_2}{G_1} \bar{f}_7(z) \right] T + \left[f_{16}(z_4) \quad - \frac{G_2}{G_1} f_8(z_4) \right] \bar{T} \right\},$$

$$w = -\frac{H}{\pi} \sum_{i=1}^{3} C_i \alpha_{i1} \left\{ \left[\bar{f}_1(z_i) + \frac{G_2}{G_1} \bar{f}_9(z_i) \right] T + \left[f_1(z_i) + \frac{G_2}{G_1} f_9(z_i) \right] \bar{T} \right\},$$

$$\Phi = -\frac{H}{\pi} \sum_{i=1}^{3} C_i \alpha_{i2} \left\{ \left[\bar{f}_1(z_i) + \frac{G_2}{G_1} \bar{f}_9(z_i) \right] T + \left[f_1(z_i) + \frac{G_2}{G_1} f_9(z_i) \right] \bar{T} \right\},$$

$$\sigma_1 = -\frac{2H}{\pi} \sum_{i=1}^{3} C_i [(c_{66} - c_{11}) + c_{13} s_i \alpha_{i1} + e_{31} s_i \alpha_{i2}] \left\{ \left[\bar{f}_5(z_i) + \frac{G_2}{G_1} \bar{f}_{10}(z_i) \right] T + \left[f_5(z_i) + \frac{G_2}{G_1} f_{10}(z_i) \right] \bar{T} \right\},$$

$$\sigma_2 = \frac{2c_{66}H}{\pi}\sum_{i=1}^{3} C_i\left\{\left[f_5(z_i) + \frac{G_2}{G_1}\bar{f}_{13}(z_i)\right]T + \left[f_{11}(z_i) + \frac{G_2}{G_1}f_{12}(z_i)\right]\bar{T}\right\}$$

$$-\frac{s_4}{\pi^2}\left\{\left[-f_5(z_4) + \frac{G_2}{G_1}\bar{f}_{13}(z_4)\right]T + \left[f_{11}(z_4) - \frac{G_2}{G_1}f_{12}(z_4)\right]\bar{T}\right\},$$

$$\sigma_z = -\frac{H}{\pi}\sum_{i=1}^{3} C_i\gamma_{1i}\left\{\left[\bar{f}_5(z_i) + \frac{G_2}{G_1}\bar{f}_{10}(z_i)\right]T + \left[f_5(z_i) + \frac{G_2}{G_1}f_{10}(z_i)\right]\bar{T}\right\},$$

$$\tau_z = \frac{H}{\pi}\sum_{i=1}^{3} C_i\gamma_{1i}s_i\left\{\left[f_3(z_i) + \frac{G_2}{G_1}\bar{f}_{14}(z_i)\right]T + \left[-f_4(z_i) + \frac{G_2}{G_1}f_{15}(z_i)\right]\bar{T}\right\}$$

$$+\frac{1}{2\pi^2}\left\{\left[f_3(z_4) - \frac{G_2}{G_1}\bar{f}_{14}(z_4)\right]T + \left[f_4(z_4) + \frac{G_2}{G_1}f_{15}(z_4)\right]\bar{T}\right\},$$

$$D = \frac{H}{\pi}\sum_{i=1}^{3} C_i\gamma_{2i}s_i\left\{\left[f_3(z_i) + \frac{G_2}{G_1}\bar{f}_{14}(z_i)\right]T + \left[-f_4(z_i) + \frac{G_2}{G_1}f_{15}(z_i)\right]\bar{T}\right\}$$

$$+\frac{e_{15}}{2\pi^2 c_{44}}\left\{\left[f_3(z_4) - \frac{G_2}{G_1}\bar{f}_{14}(z_4)\right]T + \left[f_4(z_4) + \frac{G_2}{G_1}f_{15}(z_4)\right]\bar{T}\right\}$$

and

$$D_z = -\frac{H}{\pi}\sum_{i=1}^{3} C_i\gamma_{2i}\left\{\left[\bar{f}_5(z_i) + \frac{G_2}{G_1}\bar{f}_{10}(z_i)\right]T + \left[f_5(z_i) + \frac{G_2}{G_1}f_{10}(z_i)\right]\bar{T}\right\}, \tag{26}$$

where f_1–f_5 are given in pages 235–236 and f_7–f_{16} are given in pages 247–249, respectively, in Fabrikant (1989). To save space in this paper, they are not repeated here. It is noted here that these expressions are all in terms of elementary functions, that is to say, the solution of a penny-shaped crack in a transversely isotropic piezoelectric medium subjected to point shear loading is exactly obtained in elementary functions.

Eq. (26) gives τ_z at $z = 0$ for $\rho > a$:

$$\tau_z\big|_{z=0} = \frac{1}{\pi^2 G_1}\left[G_1 f_3(0)T + G_2 f_{15}(0)\bar{T}\right], \tag{27}$$

where $f_3(0)$ and $f_{15}(0)$ can be calculated from Eqs. (4.2.9) and (4.4.50), respectively, in Fabrikant (1989), as follows

$$f_3(0) = \frac{\left(a^2 - \rho_0^2\right)^{1/2}}{R^2\left(\rho^2 - a^2\right)^{1/2}}$$

and

$$f_{15}(0) = \frac{\left(a^2 - \rho_0^2\right)^{1/2}}{\left(\rho^2 - a^2\right)^{1/2}}\frac{\left(3\rho^2 - \rho\rho_0 e^{i(\phi-\phi_0)}\right)e^{i2\phi}}{\left(\rho^2 - \rho\rho_0 e^{i(\phi-\phi_0)}\right)^2}. \tag{28}$$

Table 1
Values of G_1 and G_2 for different piezoelectric materials

Materials	References	G_1 (10^{-12} m^2/N)	G_2 (10^{-12} m^2/N)
PZT-4	Dunn and Taya (1994)	8.6144	2.7585
PZT-5	Dunn and Taya (1994)	10.798	3.7142
PZT-7A	Dunn and Taya (1994)	8.0470	2.4941
$BaTiO_3$	Dunn and Taya (1994)	6.1322	1.2724
PZT-6B	Wang and Zheng (1994)	6.8817	1.4392

The complex stress intensity factor defined by Eq. (22) can thus be obtained for point loading as follows:

$$k_{\mathrm{II}} + \mathrm{i}k_{\mathrm{III}} = \frac{\left(a^2-\rho_0^2\right)^{1/2}}{\pi^2\sqrt{2a}}\left[\frac{T\mathrm{e}^{-\mathrm{i}\phi}}{\rho_0^2+a^2-2a\rho_0\cos(\phi-\phi_0)} + \frac{G_2}{G_1}\frac{\left(3a-\rho_0\mathrm{e}^{\mathrm{i}(\phi-\phi_0)}\right)\bar{T}\mathrm{e}^{\mathrm{i}\phi}}{a\left(a-\rho_0\mathrm{e}^{\mathrm{i}(\phi-\phi_0)}\right)^2}\right]. \tag{29}$$

It can be shown that the complex intensity factor here has an identical form to that for elasticity, see Eq. (4.4.58) in Fabrikant (1989). It is pointed out that Eq. (4.4.58) there contains a minor misprint, i.e. the symbol ρ in the denominator of the first term should be ρ_0. The corresponding intensity factor for an arbitrarily distributed loading can obviously obtained by integrating Eq. (29):

$$\begin{aligned} k_{\mathrm{II}} + \mathrm{i}k_{\mathrm{III}} = \frac{1}{\pi^2\sqrt{2a}}\int_0^{2\pi}\int_0^a\left(a^2-\rho_0^2\right)^{1/2}\Bigg\{&\frac{\mathrm{e}^{-\mathrm{i}\phi}\tau(\rho_0,\phi_0)}{\rho_0^2+a^2-2a\rho_0\cos(\phi-\phi_0)} \\ &+\frac{G_2}{G_1}\frac{\left(3a-\rho_0\mathrm{e}^{\mathrm{i}(\phi-\phi_0)}\right)\mathrm{e}^{\mathrm{i}\phi}\bar{\tau}(\rho_0,\phi_0)}{a\left(a-\rho_0\mathrm{e}^{\mathrm{i}(\phi-\phi_0)}\right)^2}\Bigg\}\rho_0\,\mathrm{d}\rho_0\,\mathrm{d}\phi_0. \end{aligned} \tag{30}$$

For example, utilizing the following integrals:

$$\int_0^{2\pi}\int_0^a\frac{(a^2-\rho^2)^{1/2}}{a^2+\rho^2-2a\rho\cos\phi}\rho\,\mathrm{d}\rho\,\mathrm{d}\phi = 2\pi a$$

and

$$\int_0^{2\pi}\int_0^a\frac{(a^2-\rho^2)^{1/2}(3a-\rho\mathrm{e}^{\mathrm{i}\phi})}{(a-\rho\mathrm{e}^{\mathrm{i}\phi})^2}\rho\,\mathrm{d}\rho\,\mathrm{d}\phi = 2\pi a^2, \tag{31}$$

the complex intensity factor for a uniform loading, i.e. Eq. (25), is reproduced.

As a numerical example, we consider here the distributions of the shear displacements u and v at the crack face, i.e. at $z = 0$, $\rho < a$, due to the concentrated shear force T applied at the point (ρ_0, ϕ_0, 0), $\rho_0 < a$. This is of practical interest because u and v at the crack face are directly related to the relative slip displacements in the modes II and III crack problems. The expression of the complex displacement at $z = 0$ for $\rho < a$ corresponding to T can be directly obtained from Eq. (18) as

$$U(\rho,\phi,0) = \frac{G_1}{\pi}\left[\frac{1}{R}\tan^{-1}\left(\frac{\eta}{R}\right) - \frac{G_2^2}{G_1^2}\frac{(3-\bar{t})\eta}{a^2(1-\bar{t})^2}\right]T + \frac{G_2}{\pi}\left[\frac{q}{R\bar{q}}\tan^{-1}\left(\frac{\eta}{R}\right) + \frac{\eta(q/\bar{q}-t\mathrm{e}^{\mathrm{i}2\phi_0})}{a^2(1-t)(1-\bar{t})}\right]\bar{T}. \tag{32}$$

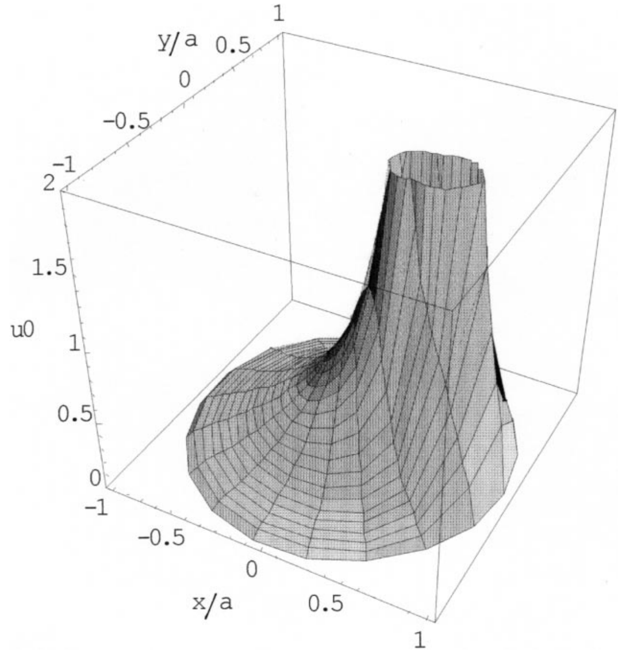

Fig. 1. Distribution of the nondimensional shear displacement $u0$ at the penny-shaped crack face due to the shear stress T_x applied at $(0.5a, 0, 0)$.

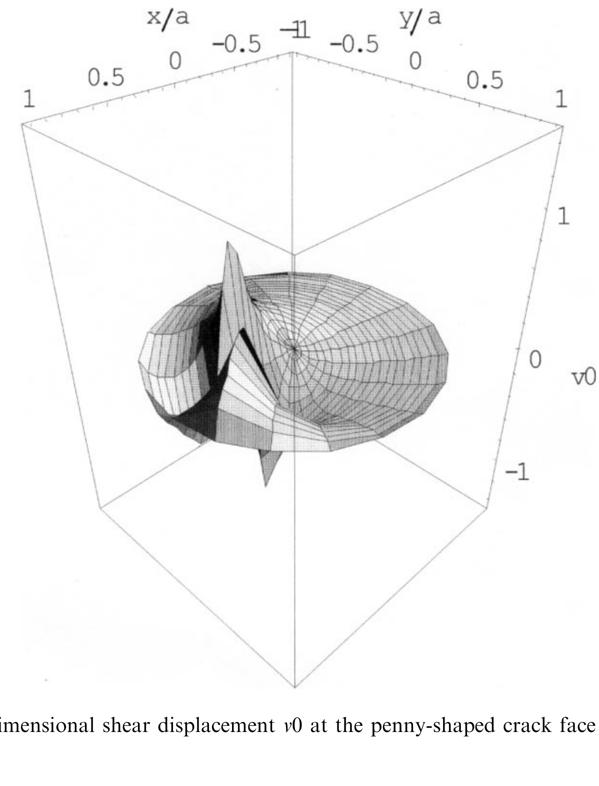

Fig. 2. Distribution of the nondimensional shear displacement $v0$ at the penny-shaped crack face due to the shear stress T_x applied at $(0.5a, 0, 0)$.

It can be seen that only two material constants, G_1 and G_2 are involved in Eq. (32). Table 1 gives their values for several piezoelectric materials.

For a numerical example, the concentrated shear force T is assumed to be applied along the x-axis at the point (0.5a, 0, 0) so that we have $T = \bar{T} = T_x$ in Eq. (32). The distributions of the nondimensional shear displacements $u0 = au/(G_1 T)$ and $v0 = av/(G_2 T)$ at the crack face are displayed in Figs. 1 and 2, respectively. The piezoelectric material is assumed to be PZT-4, for which the ratio between G_1 and G_2 is $\alpha = G_2/G_1 = 0.3202$. The singularity of the shear displacement $u0$ at the point the concentrated shear force is applied is clearly shown in Fig. 1.

5. Verification of the present method

Kogan et al. (1996) have derived exact solutions of a penny shaped crack subjected to axisymmetric as well as antisymmetric far field uniform loadings using a limiting procedure from the solutions of an elliptical inclusion problem. Now for antisymmetric uniform loading, we can compare our results with those obtained by them. Let us check the expression for the shear stress τ_{xz} at $z = 0$, $\phi = 0$ for $\rho > a$ when the crack is subjected to an out-of-plane uniform shear loading τ_{xz}^0 applied at infinity. By using the theorem of superimposition, Eq. (24) gives the expression of τ_{xz} at $z = 0$, $\phi = 0$ for $\rho > a$ due to τ_{xz}^0 as follows

$$\tau_{xz} = \tau_z\big|_{z=0} = \tau_{xz}^0 + \frac{2}{\pi G_1}\left\{G_1\left[\frac{a}{(\rho^2 - a^2)^{1/2}} - \sin^{-1}\left(\frac{a}{\rho}\right)\right] + G_2\frac{a^3}{\rho^2(\rho^2 - a^2)^{1/2}}\right\}\tau_{xz}^0. \tag{33}$$

The above equation is further converted to the following form

$$\begin{aligned}\tau_{xz} = \tau_{xz}^0&\left\{\frac{2a}{\pi(\rho^2 - a^2)^{1/2}} + \frac{2}{\pi}\tan^{-1}\left[\frac{(\rho^2 - a^2)^{1/2}}{a}\right] + \frac{2a^3}{\pi\rho^2(\rho^2 - a^2)^{1/2}}\right\}\\ &+ \frac{2(G_2 - G_1)\tau_{xz}^0}{\pi G_1}\frac{a^3}{\rho^2(\rho^2 - a^2)^{1/2}}.\end{aligned} \tag{34}$$

On the other hand, Eq. (38) in Kogan et al. (1996) reads (in our notation),

$$\begin{aligned}\tau_{xz}^{\text{Kogan et al.}} = \tau_{xz}^0&\left\{\frac{2a}{\pi(\rho^2 - a^2)^{1/2}} + \frac{2}{\pi}\tan^{-1}\left[\frac{(\rho^2 - a^2)^{1/2}}{a}\right] + \frac{2a^3}{\pi\rho^2(\rho^2 - a^2)^{1/2}}\right\}\\ &+ \mathrm{i}6 s_4 c_{44} A_4\frac{a^3}{\rho^2(\rho^2 - a^2)^{1/2}},\end{aligned} \tag{35}$$

where A_4 and other three constants, A_i ($i = 1, 2, 3$), are determined by Eq. (37) in Kogan et al. (1996), which are also rewritten as follows:

$$\sum_{i=1}^{3}\gamma_{1i}A_i = 0,$$

$$\sum_{i=1}^{3} \gamma_{2i} A_i = 0,$$

$$\sum_{i=1}^{3} \gamma_{1i} s_i A_i \bigg/ c_{44} + A_4 s_4 = -\tau_{xz}^0 \bigg/ \left(\frac{3}{2} c_{44} \pi \mathrm{i}\right)$$

and

$$\sum_{i=1}^{3} A_i = A_4, \tag{36}$$

It is seen that if

$$\frac{2(G_2 - G_1)\tau_{xz}^0}{\pi G_1} = \mathrm{i}6 s_4 c_{44} A_4, \tag{37}$$

Eqs. (34) and (35) are identical. Because $A_i = A_4 C_i$ (i = 1, 2, 3), the third equation in Eq. (36) becomes

$$\frac{1}{H} + \frac{1}{\beta} = -\frac{4\tau_{xz}^0}{3A_4\mathrm{i}}, \tag{38}$$

which finally leads to Eq. (37). Thus we have proved $\tau_{xz} = \tau_{xz}^{\text{Kogan et al.}}$. Kogan et al. (1996) also derived the modes II and III intensity factors due to τ_{xz}^0, as follows

$$(k_{\mathrm{II}} + \mathrm{i}k_{\mathrm{III}})^{\text{Kogan et al.}} = \sqrt{\frac{a}{2}}\frac{1}{\pi}\left(4\tau_{xz}^0 \cos\phi + \mathrm{i}6 s_4 c_{44} A_4 \pi \mathrm{e}^{\mathrm{i}\phi}\right). \tag{39}$$

By virtue of Eq. (37), it is easy to verify that Eq. (39) is in agreement with Eq. (25).

6. Conclusions

By employing the potential theory method as well as the recently proposed general solution, the problem of a penny-shaped crack in a transversely isotropic piezoelectric medium subjected to antisymmetric shear loading has been analyzed exactly. Complete and exact expressions for the elastoelectric field are presented for the crack subjected to uniform as well as point shear loadings. The complex stress intensity factors (mode II and mode III) are also derived in an exact manner. Numerical results are presented for point shear loading case that show clearly the singularity of shear displacement u at the point where the concentrated shear force $T = T_x$ is applied.

The correctness of the present results and the effectiveness of our method are demonstrated in the paper by comparing the results for the uniform loading with those obtained by Kogan et al. (1996).

It is worth pointing out here again that the general solution shall take other forms for equal eigenvalues. The succeeding derivations are similar to what have been described above (see Appendix A). However, as pointed out by Fabrikant (1989), one can also derive the corresponding results of equal eigenvalues directly from the ones of distinct eigenvalues, by utilizing the well-known L'Hospital rule.

Recently, Fabrikant (1996a, 1996b) also derived a complete solution of the problem of an external circular crack in a transversely isotropic elastic material subjected to arbitrary shear loading. As what have been done in this paper, it is not difficult to use his results to obtain the corresponding complete solution for transversely isotropic piezoelectric materials.

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Appendix

The general solutions for cases of multiple roots of s_i (i = 1, 2, 3) are also expressed in terms of quasi-harmonic functions F_i (i = 1, 2, 3, 4) that satisfy Eq. (6). They can be rewritten in the following complex forms (Ding et al., 1997; Chen, 2000):

Case (1): $s_1 \neq s_2 = s_3$

$$U = \Lambda(F_1 + F_2 + z_2 F_3 + \mathrm{i}F_4),$$

$$w = \alpha_{11}\frac{\partial F_1}{\partial z_1} + \alpha_{21}\frac{\partial F_2}{\partial z_2} + \alpha_{21} z_2 \frac{\partial F_3}{\partial z_2} + \alpha_{41} F_3$$

and

$$\Phi = \alpha_{12}\frac{\partial F_1}{\partial z_1} + \alpha_{22}\frac{\partial F_2}{\partial z_2} + \alpha_{22} z_2 \frac{\partial F_3}{\partial z_2} + \alpha_{42} F_3, \tag{A1}$$

where

$$\alpha_{41} = \frac{2\left(2c_{44}\epsilon_{33}s_2^2 - m_3\right)s_2 - \left(m_1 - 3m_2 s_2^2\right)\alpha_{21}}{m_1 - m_2 s_2^2}$$

and

$$\alpha_{42} = \frac{2\left(2c_{44}e_{33}s_2^2 - m_4\right)s_2 - \left(m_1 - 3m_2 s_2^2\right)\alpha_{22}}{m_1 - m_2 s_2^2}.$$

Case (2): $s_1 = s_2 = s_3$

$$U = \Lambda\left(F_1 + z_1 F_2 + z_1^2 \frac{\partial F_3}{\partial z_1} + \mathrm{i}F_4\right),$$

$$w = \alpha_{11}\left(\frac{\partial F_1}{\partial z_1} + z_1\frac{\partial F_2}{\partial z_1} + z_1^2\frac{\partial^2 F_3}{\partial z_1^2}\right) + \alpha_{41}\left(F_2 + 2z_1\frac{\partial F_3}{\partial z_1}\right) + \alpha_{51}F_3$$

and

$$\Phi = \alpha_{12}\left(\frac{\partial F_1}{\partial z_1} + z_1\frac{\partial F_2}{\partial z_1} + z_1^2\frac{\partial^2 F_3}{\partial z_1^2}\right) + \alpha_{42}\left(F_2 + 2z_1\frac{\partial F_3}{\partial z_1}\right) + \alpha_{52}F_3, \tag{A2}$$

where

$$\alpha_{51} = \frac{2\left[3m_2(\alpha_{11}+\alpha_{41})s_1^2 - m_1\alpha_{41} + \left(6c_{44}\epsilon_{33}s_1^2 - m_3\right)s_1\right]}{m_1 - m_2 s_1^2}$$

and

$$\alpha_{52} = \frac{2\left[3m_2(\alpha_{12}+\alpha_{42})s_1^2 - m_1\alpha_{42} + \left(6c_{44}e_{33}s_1^2 - m_4\right)s_1\right]}{m_1 - m_2 s_1^2}.$$

For Case (1), we assume

$$F_i(z) = c_i\left[\Lambda\bar{F}(z_i) + \bar{\Lambda}F(z_i)\right],\ (i = 1,2)$$

$$F_3(z) = c_3\left[\Lambda\bar{G}(z_3) + \bar{\Lambda}F(z_3)\right]$$

and

$$F_4(z) = c_4\left[\Lambda\bar{F}(z_4) - \bar{\Lambda}F(z_4)\right], \tag{A3}$$

where F has been given by Eq. (11) in the paper, and,

$$G(\rho,\phi,z) = \int\int_S \frac{U(N)}{R(M,N)}\,\mathrm{d}S_N. \tag{A4}$$

For Case (2), we assume

$$F_1(z) = c_1\left[\Lambda\bar{F}(z_1) + \bar{\Lambda}F(z_1)\right],$$

$$F_i(z) = c_i\left[\Lambda\bar{G}(z_i) + \bar{\Lambda}G(z_i)\right],\ (i = 2,3)$$

and

$$F_4(z) = c_4\left[\Lambda\bar{F}(z_4) - \bar{\Lambda}F(z_4)\right]. \tag{A5}$$

The followed derivatives are similar to those described in the text and omitted here for the sake of simplicity. In fact, for both cases, the resulting integro–differential equations have the same structure as Eq. (16) except for the involved constants. Therefore, previous results in potential theory are still valid to obtain the corresponding solutions.

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